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CRITERIA FOR THE FUNCTIONAL CONTROLLABILITY AND INVERTIBILITY OF NON-LINEAR SYSTEMS†

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In the development of investigations on inverse problems [1, 2], criteria for the functional controllability and invertibility of non-linear systems of equations with an output are obtained. The solution is based on the construction of an inverse system for which the input action of the initial system is the output. An identification problem is considered which corresponds to the problem of invertibility with an unknown initial state. The properties of λ-invertibility and λ-identifiability, which arise in cases when the output signal is known in a set of trajectories, are investigated.

1. FORMULATION OF THE PROBLEM

The non-linear system of equations with an output

$$\dot{x} = f(t, x, u) \tag{1.1}$$

$$y = h(t, x, u) \tag{1.2}$$

is considered.

Here, $x \in D \subseteq R^n$ is a phase vector, $u \in U \subseteq R^m$ is the input action or the control vector, which is a function of time t when $t \in T = [0, t_1] \subseteq [0, \infty)$, and $y \in Y \subseteq R^k$ is the output vector or the function being measured. It is assumed that the functions f, h and u are differentiable a sufficient number of times.

The direct and inverse control problems differ depending on whether the initial action is sought from the need to achieve an output with the required properties or the equation which achieves it is determined using a specified output. The problem of functional controllability considered below, which involves the attainment of an arbitrary output signal (possibly, differentiable a sufficient number of times), is a direct problem. The problem of invertibility, which in an inverse problem, consists of finding the initial action using a specified output. An inverse system, for which the input action of the initial system is the output, and the output signal (and its derivatives) of the initial system is the input, can be successfully used to solve both problems.

2. THE INVERSE SYSTEM

We will formulate the problem of finding the input action for a specified output signal of system (1.1), (1.2) as the output of a certain system of differential equations, the input of which would be the specified output and, possibly, its derivatives. This problem is assumed to be solved using the following scheme, which consists of several steps. We first calculate the derivatives

$$y_i^{(s_i)} = h_{is_i}(t, x, u), \quad i = 1, \ldots, k \tag{2.1}$$

which explicitly contain a parameter u. Here, derivatives of the order of $s_i - 1$ do not depend explicitly on the parameter u. Suppose that, among the functions (2.1), which are considered as functions of the variable u, there are $k_1 \leqslant k$ independent functions $y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}$, to fix our ideas, and that the remaining functions depend on them. On solving the equations $y_i^{(s_i)} = h_{is_i}(t, x, u)$ $(i = 1, \ldots, k_1)$ for $u_1, \ldots, u_{k1}$ (when necessary, the numbering u_i can be changed) and substituting the values found into the remaining equations of (2.1) and system (1.1), we obtain

$$u_i = \varphi_i(t, x, y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}, u_{k_1+1}, \ldots, u_m), \quad i = 1, \ldots, k_1 \tag{2.2}$$

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$$y_j^{(s_j)} = \Phi_{js_j}(t, x, y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}), \quad j = k_1 + 1, \ldots, k \tag{2.3}$$

$$\dot{x} = f_1(t, x, y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}, \ u_{k_1+1}, \ldots, u_m) \tag{2.4}$$

Here, in formulae (2.3) and (2.4), the functions $y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}$ are treated as functions of time t, that is, t and x are the arguments of the functions Φ_{js_j} and $t, x, u_{k_1+1}, \ldots, u_m$. are the arguments of the function f_1. We assume that, as a result of the procedure for eliminating the variables $u_1, \ldots, u_{k_1}$, the functions Φ_{js_j}, f_1 obtained are continuously differentiable functions of their arguments a sufficient number of times.

Passing to the second step, we consider system (2.4) as the initial system, for which the vector $(u_{k_1+1}, \ldots, u_m)$ is the input and the functions (2.3) are the output and we repeat the transformations of the first step. The required system is obtained when, at the next step, the derivatives of the functions (2.3) do not contain the input variables u_i. After a finite number of steps (no greater than m) we obtain

$$w = \varphi(t, x, y, \ldots, y^{(s)}, \upsilon) \tag{2.5}$$

$$\Phi_j(t, y, \ldots, y^{(s_j)}) = 0, \quad j = 1, \ldots, \varkappa \tag{2.6}$$

$$\dot{x} = \Phi(t, x, y, \ldots, y^{(s)}, \upsilon) \tag{2.7}$$

where $u = (w, \upsilon)$, $w \in W \subseteq R^{k-\varkappa}$, $\upsilon \in V \subseteq R^{m+\varkappa-k}$ and, where necessary, the numbering of the input variables can be changed. The functions Φ_j are independent and the values of s_j in formulae (2.6) are the minimum values for which these relations are possible. This agreement is necessary since differentiation of relation (2.6) with respect to t leads to a further equality with increased values of s_j, which implies an ambiguity in determining the magnitude of $\varkappa$.

We shall call system (2.5)–(2.7) the inverse system (IS) with respect to the given system (1.1), (1.2). The quantity $\varkappa$ is referred to as the output defect and $s_0 = \max(s, s_1, \ldots, s_\varkappa)$ is the smoothness index of the output of system (1.1), (1.2). Note that the output signal and its derivatives $y, \ldots, y^{(s)}$ and some of the components of the input signal υ of the initial system is the input of the IS and that some of the components of the input signal w of the initial system are the output. Here, unlike the initial system, the input action of the IS cannot be arbitrary but satisfies the differential relations (2.6). It follows from the above arguments that the form of the IS depends on the choice of the components of the input variable of the initial system, which are the output of the IS.

Remark 1. The IS can be directly introduced by defining it as the system of equations (2.5), (2.7) which reduce to identities for any solution of system (1.1), (1.2) and, conversely, any solutions of system (2.5), (2.7) from the domain of its definition reduce Eqs (1.1) and (1.2) to identities. With such a definition, the question of the existence of an IS remains open. In a number of cases, the proposed scheme for constructing the IS is globally feasible. In a local formulation, subject to the assumptions which have been made, its use always leads to the construction of the IS. Here, localness is to be understood in the sense that a property being studied holds in a certain neighbourhood belonging to the domain in which system (1.1), (1.2) is considered. Situations may arise in a local treatment at a point (all the neighbourhoods being considered contain the initial point) when it is necessary to carry out preliminary procedures concerned with the resolution of singularities.†

For the proof, we consider the first step and, for the functions (2.1) in the domain $\tau_0 \times D_0 \times U_0 \subseteq T \times D \times U$, we find

$$\max_{(t,x,u)\in\tau_0\times D_0\times U_0} \operatorname{rank} \frac{\partial(y_1^{(s_1)}, \ldots, y_k^{(s_k)})}{\partial(u_1, \ldots, u_m)} = k_1$$

Then, by virtue of the assumption which we made regarding the differentiability of the functions being considered, a neighbourhood $\tau_0^* \times D_0^* \times U_0^* \subseteq \tau_0 \times D_0 \times U_0$ exists in which the rank of the Jacobian matrix being considered is constant and equal to k_1. According to the implicit function theorem, this ensures the existence and differentiability of the functions (2.2) and (2.3) for values $t, x_1, \ldots, x_n, y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}, u_{k_1+1}, \ldots, u_m$ which belong to a certain neighbourhood $\tau_{01} \times D_{01} \times Y_1 \times U_1$, where the neighbourhood U_1 is included in the projection of the neighbourhood U_0 on the subspace of the variables $u_{k_1+1}, \ldots, u_m$. Suppose that, when $(t, x) \in \tau_1 \times D_1 \subseteq (\tau^*_0 \times D_0^*) \cap (\tau_{01} \times D_{01})$, the quantities $y_1^{(s_1)}, \ldots, y_{k_1}^{(s_{k1})}$ belong to the neighbourhood Y_1. Then, the functions Φ_{js_j}, f_1 in formulae (2.3) and (2.4), which are treated as functions of the arguments $t, x_1, \ldots, x_n, u_{k_1+1}, \ldots, u_m$, will

†The referee drew my attention to the possibility of such situations occurring.

be differentiable a sufficient number of times in the neighbourhood $\tau_1 \times D_1 \times U_1$ and, for system (2.3), (2.4), it is possible in the second step to repeat the above arguments in the new domain $\tau_1 \times D_1 \times U_1$ with reduced dimension of the equation. These conclusions also hold when the subsequent steps are carried out, which ensures the existence of the IS (2.5)–(2.7).

Remark 2. The definition of an IS introduced above and its use differ from that adopted in the literature [3–5], where an IS is defined under the assumption of the invertibility of the initial system and therefore has a special form and is used to represent the input signal in control problems when developing diverse computational procedures. In this paper, an IS is introduced for a system of general form and is used to investigate the property of the solvability of inverse control problems.

Remark 3. The derivatives of the functions (2.3) can be calculated implicitly, by differentiating the output (1.2) a sufficient number of times. In this case, the corresponding derivatives $y^{(i)}(t)$ will now depend on the derivatives of the input $u^{(i)}(t)$. Using these relations, it is possible to study the properties of an IS without its direct construction and to obtain the conditions for diverse inverse problems to be solvable, which is done in Section 5.

3. FUNCTIONAL CONTROLLABILITY AND INVERTIBILITY

The IS constructed above enables one to study the properties of functional controllability and invertibility. We shall adopt the following definition of functional controllability.

Definition 1. System (1.1) is called a functionally controlled system with respect to the output (1.2) of smoothness s at the point $x_0 \in D$ if, for any function $y(t) \in C^s(Y)$ such that

$$y^{(i)}(t_0) \in \{h_i(t_0, x_0, u(t_0)) : u(t_0) \in U\}, \quad i = 0, 1, \ldots, s$$

a control $u(t) \in U$ can be found such that $y(t) = h(t, x(t, t_0, x_0, u), u(t))$. If certain neighbourhoods of zero and the point $y(t_0)$ are taken as the domains U and Y and the function $y(t)$ is defined in a small interval $\tau_0 \subset T$ of the point t_0, then one speaks of local functional controllability.

Theorem 1. System (1.1) is a functionally controlled system with respect to the output (1.2) if and only if the output defect is equal to zero. The smoothness of the output is determined when constructing the IS.

Proof. When the conditions of the theorem are satisfied, the IS has the form of (2.5), (2.7) and no constraints whatsoever are imposed on the output $y(t)$, apart from the initial constraints: $y^{(i)}(t_0) \in \{h_i(t_0, x_0, u(t_0)) : u(t_0) \in U\}$ $(i = 0, 1, \ldots, s_0)$. System (2.7) corresponds to any specified function $y(t)$ with permissible initial values and, by solving this system with the initial condition $x(t_0) = x_0$ and any permissible control $\tilde{v}(t)$, we find $\tilde{x}(t)$. Using formulae (2.5), the solution $\tilde{x}(t)$ determines the control

$$\begin{aligned} \tilde{u}_i(t) &= \varphi_i(t, \tilde{x}(t), y(t), \ldots, y^{(s)}(t), \tilde{v}(t)), \quad i = 1, \ldots, k \\ \tilde{u}_{j+k}(t) &= \tilde{v}_j(t), \quad j = 1, \ldots, m-k \end{aligned} \tag{3.1}$$

By construction, the solution of system (1.1) with the initial condition $x(t_0) = x_0$, which corresponds to control (3.1), satisfies the relation $y(t) = h(t, x(t, t_0, x_0, \tilde{u}), \tilde{u}(t))$ which proves the sufficiency of the conditions of the theorem.

We prove necessity by contradiction. Suppose that system (1.1) is a functionally controlled system and, contrary to the assertion of the theorem, the output defect is non-zero. At least one relation of the form of (2.6) then exists

$$\Phi_1(t, y, \ldots, y^{(s)}) = 0 \tag{3.2}$$

which denotes the dependence of one of the components of the signal, $y_k(t)$, say, on the remaining components. It follows from this that, in the case of a function $y(t)$ which does not satisfy relation (3.2), the control $u(t)$ which produces this signal does not exist, that is, system (1.1) is not a functionally controlled system, which contradicts the assumption and proves the theorem.

We will now study the property of invertibility, adopting the following definition.

Definition 2. System (1.1) is said to be invertible with respect to the output (1.2) at a point $x_0 \in D$ if, for any two different permissible functions $u_1(t)$, $u_2(t)$, an instant $t \in T$ exists such that

$$h(t, x(t, t_0, x_0, u_1), u_1(t)) \neq h(t, x(t, t_0, x_0, u_2), u_2(t))$$

If a certain neighbourhood of zero is taken as the domain U and the functions $u(t), x(t), y(t)$ are defined in a small interval $\tau \in T$ of the point t_0, then one speaks of local invertibility.

The following theorem provides a criterion for invertibility.

Theorem 2. System (1.1) is invertible with respect to the output (1.2) at a point $x_0 \in D$ if and only if the output defect is equal to $\varkappa = k - m$, the functions φ, defined by formula (2.5), are single-valued and system (2.7) satisfies the conditions for the existence and uniqueness of a solution of the Cauchy problem.

Proof. To prove sufficiency we note that, when the conditions of the theorem are satisfied, the IS has the form

$$\dot{x} = \Phi(t, x, y, \ldots, y^{(s)}) \tag{3.3}$$

$$u = \varphi(t, x, y, \ldots, y^{(s)}) \tag{3.4}$$

By virtue of the fact that the functions (3.4) are single-valued and the uniqueness of the solution of the Cauchy problem in the case of system (3.3), the mapping $y(t) \to u(t)$ will be unique, will be unique, which implies the invertibility of system (1.1) with respect to the output (1.2).

We prove necessity by contradiction. Suppose that, in spite of the assertion, $\varkappa \neq k - m$. Since, $\varkappa \geqslant k - m$, then $\beta = \varkappa - k + m > 0$ and the IS has the form of (2.5)–(2.7) where $\dim \upsilon = \beta > 0$. For an arbitrary permissible output signal $y(t)$ and two continuous functions $\upsilon_1(t)$, $\upsilon_2(t)(\upsilon_1(t) \not\equiv \upsilon_2(t))$, the solution of the Cauchy problem for system (2.7) together with formula (2.5) determines the functions $w_1(t)$, $w_2(t)$. According to the construction of the IS, one and the same output $y(t)$ corresponds to the input signals $u_1(t) = (w_1(t), \upsilon_1(t))$, $u_2(t) = (w_2(t), \upsilon_2(t))(u_1(t) \not\equiv u_2(t)$, by virtue of the choice of the functions $\upsilon_1(t), \upsilon_2(t))$, which implies that system (1.1) is not invertible. The resulting contradiction proves the theorem.

We will now consider some examples which illustrate the use of the theorems proved above.

Example 1. We will investigate the functional controllability and invertibility of the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u_2 \tag{3.5}$$

with respect to the output

$$y_1 = x_1 + u_1 + u_2, \quad y_2 = x_2 + u_1 + u_2 \tag{3.6}$$

In the first step of the construction of the IS from Eqs (3.6) we find

$$u_1 = y_1 - x_1 - u_2 \tag{3.7}$$

$$y_2 = x_2 - x_1 + y_1 \tag{3.8}$$

In the second step we consider system (3.5) with an input u_2 and output (3.8) with a dimension which is one less compared with the initial dimensions. We have

$$\dot{y}_2 = x_3 - x_2 + \dot{y}_1, \quad \ddot{y}_2 = u_2 - x_3 + \ddot{y}_1 \tag{3.9}$$

From Eq. (3.7) and the second equation of (3.9), we obtain

$$u_1 = y_1 + \ddot{y}_1 - \ddot{y}_2 - x_1 - x_3, \quad u_2 = \ddot{y}_2 - \ddot{y}_1 + x_3 \tag{3.10}$$

Substituting the expression for u_2 into system (3.5), we have

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = \ddot{y}_2 - \ddot{y}_1 + x_3 \tag{3.11}$$

Equations (3.11) and (3.10) form the IS, for which $\varkappa = 0$ and the conditions of Theorems 1 and 2 are satisfied.

On this basis, we conclude that system (3.5) is a functionally controlled system and it is invertible with respect to the output (3.6).

Example 2. We will now consider the functional controllability and invertibility of system (3.5) with respect to the output

$$y_1 = x_1 + u_1 + u_2, \quad y_2 = x_2 + u_1 + u_2, \quad y_3 = x_3 \tag{3.12}$$

This differs from the preceding example in the fact that the output is supplemented with a third component y_3, which we use in carrying out the first step.

From the equation $\dot{y}_3 = u_2$ and the second equation of (3.12), we find

$$u_1 = y_2 - \dot{y}_3 - x_2, \quad u_2 = \dot{y}_3 \tag{3.13}$$

We now eliminate u_1, u_2 from system (3.5) and the first equation of (3.12) using formulae (3.13)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = \dot{y}_3 \tag{3.14}$$

$$y_1 = x_1 - x_2 + y_2 \tag{3.15}$$

By virtue of system (3.14), differentiating expression (3.15) we obtain the equations

$$\dot{y}_1 = x_2 - x_3 + \dot{y}_2; \quad \ddot{y}_1 = x_3 - \dot{y}_3 + \ddot{y}_2 \tag{3.16}$$

which, together with the last relation of (3.12) and Eq. (3.15), we use to obtain the condition imposed on the output variable which does not contain the phase variable

$$\ddot{y}_1 - \ddot{y}_2 - y_3 + \dot{y}_3 = 0 \tag{3.17}$$

Equations (3.14) and (3.13) form an IS for which $\varkappa = 1$. Using Theorems 1 and 2, we conclude that system (3.6) is not a functionally controlled system but it is invertible with respect to the output (3.12).

Remark 4. The equalities $\varkappa = 0$, $\varkappa = k - m$ can only be satisfied for $k \geqslant m$. It follows from this that the inequality $k \geqslant m$ is a necessary condition for functional controllability and invertibility, that is, the dimension of the output must not be less than the dimension of the input.

4. IDENTIFIABILITY AND OBSERVABILITY WITH RESPECT TO SOME OF THE VARIABLES

A version of the problem of invertibility, when the initial state $x_0 = x(t_0)$ of system (1.1) is assumed to be unknown, is important in applications. In this case, the problem of determining the initial signal is known as the problem of identification. The property of identifiability can be introduced in the following manner [1, 2].

Definition 3. System (1.1) is said to be identifiable with respect to the output (1.2) in a domain D, if, for any two different permissible functions $u_1(t)$, $u_2(t)$ and any solutions $x_1(t) \in X_{u1}, x_2(t) \in X_{u2}$, an instant of time $t \in T$ exists such that $h(t, x_1(t), u_1(t)) \neq h(t, x_2(t), u_2(t))$. Here, X_{ui} is the set of solutions of system (1.1) for $u = u_i(t)$ and any initial values of $x_0 \in D$.

The solution of the problem of identification, compared with the problem of invertibility, requires additional study due to the need to eliminate the phase variable x in formula (2.5). For this purpose, we shall analyse all the information which is obtained when constructing the IS. Together with relations (2.5) and (2.6), there are still the equations

$$\psi_\alpha(t, x, y, \ldots y^{(p_\alpha)}) = 0, \quad \alpha = 1, \ldots, \nu \tag{4.1}$$

which were used to eliminate the variable x in relations (2.6). It is obvious that, if the variable x can be eliminated from formulae (2.5) using functions (4.1), then, in the case of the invertibility of system (1.1), it will also be identifiable. It turns out that this condition is also necessary.

Theorem 3. System (1.1) is identifiable with respect to the output (1.2) in the domain D if and only if the output defect is equal to $\varkappa = k - m$, system (2.7) satisfies the conditions for the existence and uniqueness of the solution of the Cauchy problem and the functions (2.5) are single-valued and independent of x.

Proof. When the conditions of the theorems are satisfied, the system is invertible and, for a specified initial value of x_0, the input signal is found uniquely using formulae (3.4), and since these formulae are independent of x, they also give the solution of the identification problem, which proves the sufficiency of the conditions of the theorem.

To prove necessity, we note that, if the system is identifiable, then it is invertible and, when account is taken of Theorem 2, it remains to prove that functions (3.4) are independent of x. Arguing by contradiction, we assume that it is not possible to eliminate the variable x from functions (3.4) and that they do depend on x. We select the values x_{10}, x_{20} and the permissible signal $y(t)$ in such a way that $u_1(t_0) = \varphi(t_0, x_{10}, y(t_0), \ldots, y^{(s)}(t_0) \neq \varphi(t_0, x_{20}, y(t_0), \ldots, y^{(s)}(t_0)) = u_2(t_0)$. Then, the solutions $x_1(t), x_2(t)$ of the Cauchy problem for system (3.3) with $y(t)$ and the initial data x_{10}, x_{20} determine, in accordance with formulae (3.4), the two inputs $u_1(t), u_2(t)$ such that $u_1(t) \not\equiv u_2(t)$ and $h(t, x_1(t), u_1(t)) \equiv h(t, x_2(t), u_2(t)) \equiv y(t)$. This implies the non-identifiability of system (1.1) and contradicts the initial assumption. The theorem is proved.

We will use this theorem to study the property of the identifiability of system (3.5) in examples 1 and 2. In example 1, there are two equations (3.8) and (3.9) which do not contain the input variable. It is not possible to eliminate the phase variable in formulae (3.10), which define the input signal using (3.8) and (3.9) which do not contain the input variable. It is not possible to eliminate the phase variable in formulae (3.10), which define the input signal using (3.8) and (3.9). Hence, by Theorem 3, system (3.5) is non-identifiable with respect to the output (3.6). In example 2, there are three equations (the last equation of (3.12), Eq. (3.15) and the first equation of (3.16)), which can be used to eliminate the phase variable from formulae (3.13) from which, by Theorem 3, the identifiability of system (3.5) with respect to the output (3.12) follows.

Cases are encountered in identification problems when the structure of the input signal is known and, quite often, it is the solution of the system of differential equations

$$\dot{u} = g(t, x, u) \tag{4.2}$$

The problem of finding the variable u for system (1.1), (4.2) with respect to the output (1.2) when $x_0 = x(t_0)$ is unknown is an observation problem with respect to some of the variables. The property of observability with respect to some of the variables is introduced by the following definition [2].

Definition 4. System (1.1), (4.2) is said to be observable with respect to a variable u and the output (1.2) in domain D if, for any two solutions $(x_1(t), u_1(t))$, $(x_2(t), u_2(t))$ such that $u_1(t) \not\equiv u_2(t)$, an instant $t \in T$ exists such that $h(t, x_1(t), u_1(t)) \neq h(t, x_2(t), u_2(t))$.

Theorem 3 obviously gives the necessary and sufficient conditions for observability with respect to some of the variables. However, the solution of this problem can be obtained by the direct use of an augmented observation vector, consisting of the output signal and its derivatives, without constructing the IS. This enables us to obtain quite simply the sufficient conditions for observability with respect to some of the variables in a local formulation using a theorem on implicit functions.

5. SUFFICIENT CONDITIONS

With the aim of obtaining the solution of inverse problems without constructing the IS, we introduce the augmented observation vector [1, 2]

$$\begin{aligned} &y^{(0)}(t) = h_0(t, x, u) = h(t, x, u) \\ &y^{(i)}(t) = h_i(t, x, u, \dot{u}, \ldots, u^{(i)}) = \frac{\partial h_{i-1}}{\partial t} + \frac{\partial h_i}{\partial x} f(t, x, u) + \\ &+ \sum_{j=0}^{i-1} \frac{\partial h_{i-1}}{\partial u^{(j)}} u^{(j+1)}, \quad i = 1, \ldots, n \end{aligned} \tag{5.1}$$

We use the notation

$$z = (y^T,\ \dot{y}^T, \ldots, y^{(n)T})^T,\quad \mathrm{v} = (\dot{u}^T, \ldots, u^{(n)T})^T$$

$$H(t,x,u,\mathrm{v}) = (h_0^T(t,x,u), \ldots, h_n^T(t,x,u,\mathrm{v}))^T$$

and rewrite Eq. (5.1) in the form

$$z(t) = H(t,x,u,\mathrm{v}),\quad \mathrm{v} \in U_\mathrm{v} \subseteq R^{mn} \tag{5.2}$$

The choice of the domain U_v determines the class of permissible input signals in the inverse problems being considered. It follows from Theorems 2 and 3 that the uniqueness of the solution of the inverse problems is ensured by the existence of the functions $u = \varphi(t,x,z)$ or $u = \varphi(t,z)$, which are the solution of system (5.2).

The following lemma [2] gives the sufficient conditions for the existence of a solution of such a form in the case of the system of algebraic equations

$$z - F(x,y) = 0,\quad x \in P \subseteq R^n,\quad y \in Q \subseteq R^k,\quad z \in E \subseteq R^l \tag{5.3}$$

Lemma 1. Suppose that the system of equations (5.3) is given in the domain $P \times Q \times E$, $F \in C^p(P \times Q)$ and at a certain point $(x_0, y_0) \in P \times Q$

$$\operatorname{rank} \frac{\partial F(x,y)}{\partial(x,y)} = n + \operatorname{rank} \frac{\partial F(x,y)}{\partial y} \tag{5.4}$$

Then, neighbourhoods B_x, B_y, B_z of the points x_0, y_0, $z_0 = F(x_0, y_0)$ and the function $G \in C^p(B_z)$ exist such that the x coordinate of the solution (x, y) of system (5.3) is described by the formula $x = G(z)$.

On applying Lemma 1 to system (5.2), we obtain the sufficient conditions for local invertibility, identifiability and observability with respect to some of the variables from condition (5.4) using Theorems 2 and 3.

Theorem 4. Suppose that, at a certain point $(t_0, x_0, u_0, \mathrm{v}_0) \in T \times D \times U \times U_\mathrm{v}$

$$\operatorname{rank} \frac{\partial H(t,x,u,\mathrm{v})}{\partial(u,\mathrm{v})} = m + \operatorname{rank} \frac{\partial H(t,x,u,\mathrm{v})}{\partial \mathrm{v}}$$

Then, system (1.1) is locally invertible (in the neighbourhood B_u of the point u_0) with respect to the output (1.2) at the point x_0.

Theorem 5. Suppose that, at a certain point $(t_0, x_0, u_0, \mathrm{v}_0) \in T \times D \times U \times U_\mathrm{v}$

$$\operatorname{rank} \frac{\partial H(t,x,u,\mathrm{v})}{\partial(x,u,\mathrm{v})} = m + \operatorname{rank} \frac{\partial H(t,x,u,\mathrm{v})}{\partial(x,\mathrm{v})}$$

Then, system (1.1) is locally identifiable (in a certain neighbourhood $B_x \times B_u$ of the point (x_0, u_0) with respect to the output (1.2).

In a problem of observation with respect to some of the variables, when the input signal is the solution of the system of equations (4.2), it is necessary to replace formulae (5.1) and (5.2), which introduce the augmented observation vector, by the following formulae

$$y^{(0)}(t) = h_0(t,x,u) = h(t,x,u)$$
$$y^{(i)}(t) = h_i(t,x,u) = \frac{\partial h_{i-1}}{\partial t} + \frac{\partial h_{i-1}}{\partial x} f(t,x,u) + \frac{\partial h_{i-1}}{\partial u} g(t,x,u),\quad i = 1, \ldots, n \tag{5.5}$$

$$z(t) = H(t,x,u) \tag{5.6}$$

The following theorem give the sufficient conditions for local observability with respect to some of the variables.

Theorem 6. Suppose that, at a certain point $(t_0, x_0, u_0) \in T \times D \times U$

$$\operatorname{rank}\frac{\partial H(t,x,u)}{\partial(x,u)} = m + \operatorname{rank}\frac{\partial H(t,x,u)}{\partial x}$$

Then, system (1.2), (4.2) is locally observable (in a certain neighbourhood B_u of the point u_0) with respect to the variable u and the output (1.2).

6. THE USE OF SETS OF TRAJECTORIES

The formulation of identification and invertibility problems enables certain trajectories to be used which, in fact, is implemented in practical problems mainly with the aim of carrying out statistical data processing. However, such a generalization of these problems is also of theoretical significance since it extends the class of systems which possess the properties of identifiability and invertibility. Following [1, 2], we now present the corresponding definitions.

Definition 5. System (1.1) is said to be λ-invertible (invertible along λ-trajectories) with respect to the output (1.2) at a point $(x_{01}, \dots, x_{0\lambda}) \in D^\lambda$ if, for any two different permissible functions $u_1(t), u_2(t)$, an instant $t \in T$ exists such that

$$(h^T(t,x(t,t_0,x_{01},u_1),\ u_1(t)),\dots,h^T(t,x(t,t_0,x_{0\lambda},u_1),\ u_1(t)))\neq$$
$$\neq(h^T(t,x(t,t_0,x_{01},u_2),\ u_2(t)),\dots,h^T(t,x(t,t_0,x_{0\lambda},u_2),\ u_2(t)))$$

Definition 6. System (1.1) is said to be λ-identifiable (identifiable along λ-trajectories) with respect to the output (1.2) in domain D if, for any two different permissible functions $u_1(t), u_2(t)$ and any solutions $x_{11}(t), \dots, x_{1\lambda}(t) \in X_{u1}, x_{21}(t), \dots, x_{2\lambda}(t) \in X_{u2}$, an instant $t \in T$ exists such that

$$(h^T(t,x_{11}(t),\ u_1(t)),\dots,h^T(t,x_{1\lambda}(t),\ u_1(t)))\neq(h^T(t,x_{21}(t),\ u_2(t)),\dots,h^T(t,x_{2\lambda}(t),\ u_2(t)))$$

It is obvious that, if the system is invertible and identifiable, then it is λ-invertible and λ-identifiable respectively for any λ. It is therefore natural to study the properties of λ-invertibility and λ-identifiability in the case of non-invertible and non-identifiable system (along a single trajectory).

An investigation using an IS can be carried out by two methods. The first method consists of using the IS (2.5)–(2.7), constructed along a single trajectory. By Theorems 2 and 3, we conclude that, for the case being considered, the function φ in formula (2.5) contains the vector υ and, for its unique solvability, it is necessary and sufficient to be able to determine the vector υ from the relations

$$w=\varphi(t,\ x_i,\ y_i,\ \dot{y}_i,\dots,y_i^{(s)},\ v),\quad i=1,\dots,\lambda \tag{6.1}$$

In order to find υ from relations (6.1), we obtain the equivalent system

$$\varphi_{(i)}=\varphi(t,\ x_{i+1},\ y_{i+1},\dots,y_{i+1}^{(s)},\ v)-\varphi(t,\ x_1,\ y_1,\dots,y_1^{(s)},\ v)=0,\ \ i=1,\dots,\lambda-1 \tag{6.2}$$

Subject to the condition for the unique solvability of system (6.2) for υ from Eqs (6.1), the vector w is uniquely found and, together with the value of υ obtained, uniquely defines the input signal $u = (w, \upsilon)$, which implies the λ-invertibility of system (1.1) with respect to the output (1.2).

The result obtained can be formulated in the form of a theorem.

Theorem 7. System (1.1) is λ-invertible with respect to the output (1.2) if and only if it is invertible, or system (2.7) satisfies the conditions for the existence and uniqueness of the solution of the Cauchy problem, the function (2.5) is single-valued and the system of equations (6.3) is uniquely solvable for the vector υ.

In order to analyse the property of λ-identifiability, it is necessary to use Eqs (4.1) which relate the phase variable and the output signal in the trajectories being considered

$$\psi_\alpha(t,\ x_i,\ y_i,\dots,y_i^{(p_\alpha)})=0,\ \ \alpha=1,\dots,\nu;\ \ i=1,\dots,\lambda \tag{6.3}$$

If system (6.2) is uniquely solvable for υ and Eqs (6.2) and (6.3) enable one to eliminate the variables x_i from the formulae for w and υ, by obtaining them in the form

$$w = w(t,\ y_1,\dots,y_\lambda,\dots,y_\lambda^{(q)}),\ \upsilon = \upsilon(t,\ y_1,\dots,y_\lambda,\dots,y_\lambda^{(q)}) \tag{6.4}$$

then, on repeating the proof of Theorem 3, we obtain that system (1.1) is λ-identifiable with respect to the output (1.2).

Theorem 8. System (1.1) is λ-identifiable, or system (2.7) satisfies the conditions for the existence and uniqueness of a solution of the Cauchy problem, the function (2.5) is single-valued and the system of equations (6.2) and (6.3), together with formula (2.5), determine the vectors w and υ in a unique manner in the form of (6.4).

The second method involves constructing the inverse system for a system with the phase vector $X_\lambda = (x_1^T, \dots, x_\lambda^T)^T \in D^\lambda$, the output signal $Y_\lambda = (y_1^T, \dots, y_\lambda^T)^T$ and the initial input signal

$$\dot{x}_i = f(t,\ x_i,\ u),\quad i = 1,\dots,\lambda \tag{6.5}$$

$$y_i = h(t, x_i, u),\ i = 1,\dots, \lambda \tag{6.6}$$

Repeating the arguments from Section 2, we construct the IS for system (6.5) with the output (6.6). On applying Theorems 2 and 3 to it, we obtain the criteria for λ-invertibility and λ-identifiability. We obtain the sufficient conditions for λ-invertibility and λ-identifiability from Theorems 4 and 5.

Theorem 9. Suppose that, at a certain point of the domain $T \times D^\lambda \times U \times U_\nu$

$$\operatorname{rank}\frac{\partial(H(t,x_1,u,\nu),\dots,H(t,x_\lambda,u,\nu))}{\partial(u,\nu)} = m + \operatorname{rank}\frac{\partial(H(t,x_1,u,\nu),\dots,H(t,x_\lambda,u,\nu))}{\partial\nu}$$

System (1.1) is then locally λ-invertible with respect to the output (1.2) at this point.

Theorem 10. Suppose that, at a certain point of the domain $T \times D^\lambda \times U \times U_\nu$

$$\operatorname{rank}\frac{\partial(H(t,x_1,u,\nu),\dots,H(t,x_\lambda,u,\nu))}{\partial(x_1,\dots,x_\lambda,u,\nu)} = m + \operatorname{rank}\frac{\partial(H(t,x_1,u,\nu),\dots,H(t,x_\lambda,u,\nu))}{\partial(x_1,\dots,x_\lambda,\nu)}$$

System (1.1) is then locally λ-identifiable with respect to the output (1.2).

Similar results hold in the case of the problem of identifying an input signal of known structure. For instance, Theorem 10, in the formulation of which the vector ν does not occur and the function $H(t, x, u)$ is defined by formulae (5.5) and (5.6), give the sufficient conditions for local λ-identifiability. By analysing the problem of identification using a set of trajectories in the case of constant parameters u = const, it is possible to find two numbers [2]

$$\lambda_{\min} = [(m-1)\alpha^{-1}] + 1,\quad \lambda_{\max} = m + 1 - \alpha$$

$$\alpha = \operatorname{rank}\frac{\partial H(t,x,u)}{\partial(x,u)} - \operatorname{rank}\frac{\partial H(t,x,u)}{\partial x}$$

where $[\cdot]$ is the largest integer function and the function $H(t, x, u)$ is calculated using formulae (5.5) and (5.6), in which it is necessary to put $g = 0$. The following property is established using them: system (1.1) cannot be identifiable with respect to $\lambda < \lambda_{\min}$ trajectories; if system (1.1) is non-identifiable with respect to $\lambda < \lambda_{\max}$ trajectories, then it is non-identifiable with respect to any number of trajectories.

Example 3. We now consider the system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3 u_1 + u_2,\ \dot{x}_3 = u_2 \tag{6.7}$$

with the output

$$y_1 = x_1,\ y_2 = x_2 \tag{6.8}$$

In order to construct the IS, we calculate

$$\dot{y}_1 = x_2,\ \dot{y}_2 = x_3u_1 + u_2 \tag{6.9}$$

From formulae (6.8) and (6.9), we find the dependence of the input signal on the output signal and the relation for the output signal and the IS

$$w = \dot{y}_2 - x_3v,\ u = (v,\ w) \tag{6.10}$$

$$\dot{y}_1 = y_2 \tag{6.11}$$

$$\dot{x}_1 = x_2,\ \dot{x}_2 = y_2,\ \dot{x}_3 = \dot{y}_2 - x_3v \tag{6.12}$$

The existence of relation (6.11) shows that the defect $\varkappa$ of system (6.7) is equal to unity and the condition $\varkappa = k - m$ of Theorem 2 is therefore not satisfied and system (6.7) is not invertible, and this means that it is not identifiable with respect to the output (6.8).

In order to investigate λ-invertibility, we write down Eq. (6.2) for $\lambda = 2$: $\varphi_{(1)} = \dot{y}_{22} - x_{32}\upsilon - \dot{y}_{21} + x_{31}\upsilon = 0$ which is uniquely solvable for υ when $x_{32} \neq x_{31}$. We conclude on the basis of Theorem 7 that system (6.7) is λ-invertible with respect to the output (6.8) in the case when $\lambda \geqslant 2$. In order to study λ-identifiability we consider Eqs (6.3) which, in the given case, are identical to Eqs (6.8). These equations do not enable one to eliminate the variables x_{3i} from the expression for υ for any λ which indicates that the conditions of Theorem 8 are not satisfied and implies the non-identifiability of system (6.7) with respect to the output (6.8) for any number of trajectories.

We will now verify that the sufficient conditions for λ-invertibility and λ-identifiability are satisfied. From Eqs (6.9), we find

$$\det\frac{\partial(\dot{y}_{21},\ \dot{y}_{22})}{\partial(u_1,\ u_2)} = x_{31} - x_{32} \neq 0 \text{ when } x_{31} \neq x_{32}$$

The conditions of Theorem 9 are therefore satisfied for $H(y_1, y_2, \dot{y}_1, \dot{y}_2)$, $\lambda = 2$ and system (6.7) is locally λ-invertible for $\lambda \geqslant 2$. By writing out formulae (5.1) and (5.2) for the extended observation vector, it can be shown that, for any λ

$$\operatorname{rank}\frac{\partial(H(x_1,u,\mathrm{v}),\ldots,H(x_\lambda,u,\mathrm{v}))}{\partial(x_1,\ldots,x_\lambda,u,\mathrm{v})} = \operatorname{rank}\frac{\partial(H(x_1,u,\mathrm{v}),\ldots,H(x_\lambda,u,\mathrm{v}))}{\partial(x_1,\ldots,x_\lambda,\mathrm{v})}$$

which is taken to mean that the sufficient conditions for λ-identifiability are not satisfied for any λ.

Example 4. Consider the system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = x_1u_1 + u_2 \tag{6.13}$$

with the output

$$y = x_1 \tag{6.14}$$

The IS has the form

$$w = \ddot{y} - x_1v,\quad u = (v,\ w);\quad \dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = \dddot{y} \tag{6.15}$$

The conditions of Theorem 2 are not satisfied, and system (6.13) is not invertible and non-identifiable with respect to the output (6.14). Note that this conclusion follows from the fact that the dimension of the output signal is smaller than the dimension of the input signal and there is no need to construct the IS (6.15) to analyse the invertibility. However, it is possible to investigate λ-invertibility and λ-identifiability using the IS. It follows from the first formula of (6.15) that, for $\lambda = 2$, the vector υ is uniquely determined from the equation $\varphi_{(1)} = 0$ and, by Theorem 7, system (6.13) is λ-invertible with respect to the output (6.14) for $\lambda \geqslant 2$. In order to investigate λ-identifiability, we write out Eqs (6.3): $y_i = x_{1i}, \dot{y}_i = x_{2i}, \ddot{y}_i = x_{3i}$ $(i = 1, \ldots, \lambda)$, which are uniquely solvable for the phase variables x_i and enable one to eliminate them from the expressions for the vectors υ and w. It follows from Theorem 8 that system (6.13) is λ-identifiable with respect to the output (6.14) for $\lambda \geqslant 2$. By calculating the augmented observation vector using formulae (5.1) and (5.2), it can be shown that the sufficient conditions of Theorems 9 and 10 are satisfied for $\lambda \geqslant 2$.

The examples considered show that, in the case of non-linear systems, the use of a set of trajectories actually extends the possibilities for recovering the input signal, enabling one, in a number of cases, to solve a problem even when the output signal is of a lower dimension than that of the input, which is impossible in principle when a single trajectory is used. In the case of linear systems $\dot{x} = A(t)x + B(t)u$ and linear output signals $y = C(t)x + D(t)u$ the use of a set of trajectories does not offer any advantages since the IS

$$\dot{x} = A_0(t)x + B_0(t)\upsilon + Q_0(t)z; \quad w = E_0(t)x + F_0(t)z + G_0(t)\upsilon, \quad R_0(t)z = 0$$

depends linearly on the vector υ, and Eqs (6.2) are therefore independent of υ, which means that the vector υ cannot be determined from them and it therefore does not lead to λ-invertibility.

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